Section 5.4 The Fundamental Theorem of Calculus
(Minimum Homework: all odds)

We will learn how to find the exact area between the graph of a continuous (unbroken) function and the x-axis in this section.

We need to learn a new symbol to perform the area calculations in this section.

Definite integral notation / symbol.

$\int_a^b f(x)dx$

This integral notation is called a definite integral (the letters a, b make it definite)

This integral notation asks us to calculate the "net" area between the graph of $f(x)$ and the x-axis over the interval [a,b] (provided the graph of $f(x)$ is continuous over the interval)

The Fundamental theorem of Calculus

Given a function $f(x)\ that\ is\ continuous\ over\ the\ interval\ [a,b]$

$if\ F(x) = \int f(x)dx$

Then: $\int_a^b f(x)dx = F(b) - F(a)$

This theorem basically tells us that the desired area can be calculated as follows:

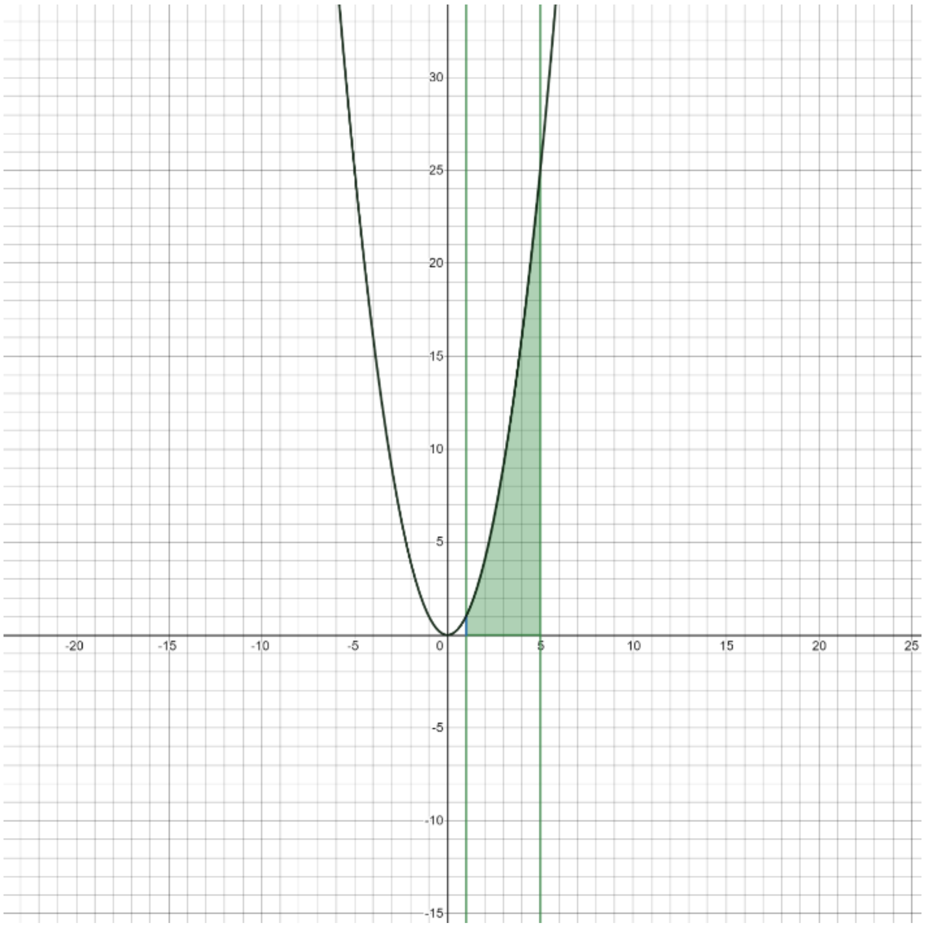
Step 1: Perform the integration

Step 2: Evaluate the integral at b, then a.

Step 3: Subtract the results to determine the area.

Here is a graph of $f(x) = x^2$ that we looked at section 5.3.

Calculating $\int_1^5 x^2 \, dx$ will give us the shaded area



Example: Use the Fundamental Theorem of Calculus to evaluate the definite integral.

$\int_1^5 x^2\,dx$

Step 1: Perform the integration. Use this rule: Power Rule:

$\int_1^5 x^2\,dx = \frac{1}{3}x^3\Big|_1^5$

(notice I kept the 1 and 5, as I will need these numbers on step 2.)
(I don't use the $+\,C$ when I am finding a definite integral. The C would cancel on the next step.)

Step 2: Evaluate the integral at 5, then at 1

$\frac{1}{3} * 5^3 = \frac{1}{3} * 125 = 125/3$

$\frac{1}{3} * 1^3 = \frac{1}{3} * 1 = 1/3$

Step 3: Subtract the results to determine the answer

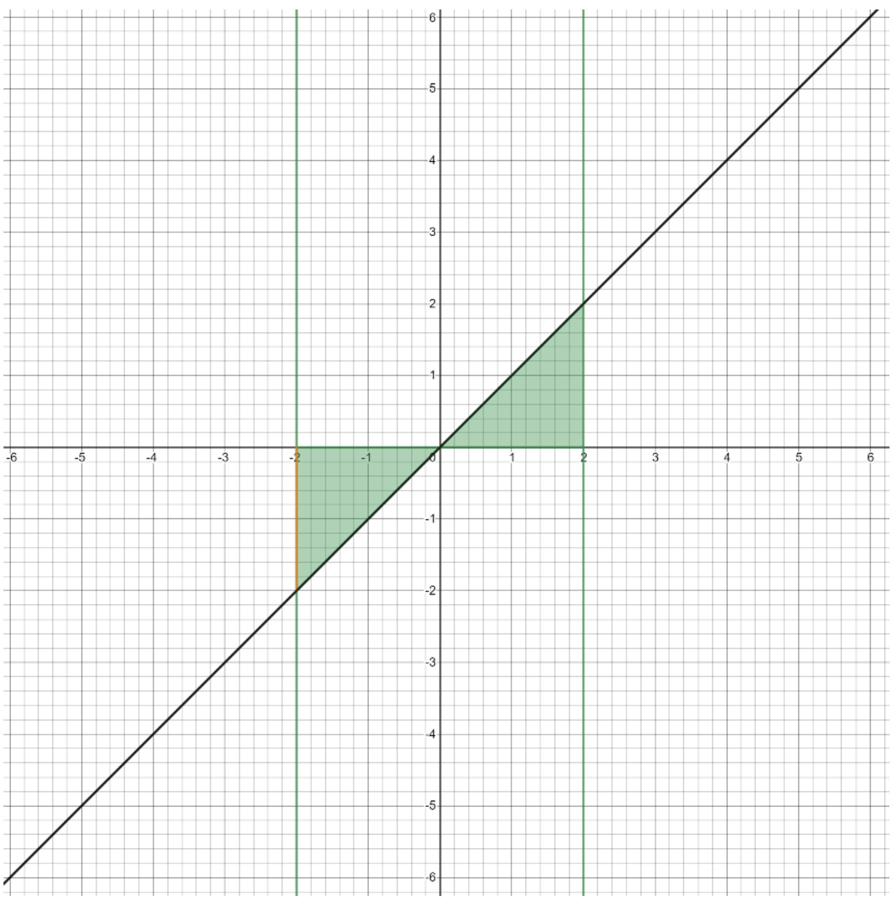
$\int_1^5 x^2\,dx = \frac{1}{3}x^3\Big|_1^5 \;=\; \frac{125}{3} - \frac{1}{3} = \frac{124}{3} \cong 41.33$

Answer: $\int_1^5 x^2\,dx = \frac{124}{3}$
(I will show you how to check this answer using a TI graphing calculator)

The area that the Fundamental Theorem of Calculus computes is a "net" area.

- areas of regions above the $x - axis$ are positive.
- areas of regions beneath the $x - axis$ are negative.

Here is a graph of the function $f(x) = x \quad with\ shading\ between - 2, and\ 2$



The graph produces 2 triangles that have the same area.
Both triangles have a base of 2 and a height of 2

Top triangle area: $\frac{2*2}{2} = 2\ square\ units$

Bottom triangle area: $\frac{2*2}{2} = 2\ square\ units$

The integral use to compute the shaded area will equal 0.

$\int_{-2}^{2} x\, dx = 0$

The bottom triangle's area will be considered -2, and the top triangle's area $+2$. The net area will be 0.

Let us find the integral to confirm that it is in fact equal to 0.

Example: Use the Fundamental Theorem of Calculus to evaluate the definite integral.

$\int_{-2}^{2} x\,dx$

Step 1: Perform the integration. Use this rule: Power Rule:
$\int x^n dx = \frac{1}{n+1} x^{n+1} + C \quad provided\ x \neq -1$

$\int_{-2}^{2} x\,dx = \frac{1}{2} x^2 |_{-2}^{2}$

Step 2: Evaluate the integral at 2, then at -2

$\frac{1}{2}(2)^2 = \frac{1}{2} * 4 = 2$

$\frac{1}{2}(-2)^2 = \frac{1}{2} * 4 = 2$

Step 3: Subtract the results to determine the answer
$\int_{-2}^{2} x\,dx = 2 - 2 = 0$

Answer: $\int_{-2}^{2} x\,dx = 0$

Example: Use the Fundamental Theorem of Calculus to evaluate the definite integral.

$\int_1^e 6x^{-1}dx$

Step 1: Perform the integration.

Rewrite using: $\int af(x)dx = a\int f(x)dx$

$\int_1^e 6x^{-1}dx$ =6 $\int_1^e x^{-1}dx$

Integrate using: "ln" Rule: $\left\{\int x^{-1}dx = \ln|x| + C\right.$

$\int_1^e 6x^{-1}dx$ =6 $\int_1^e x^{-1}dx$ = $6ln|x|_1^e$ (instead of writing a C I keep the $1\ and\ e$)

Step 2: Evaluate the integral at e, then at 1

$6ln|e| = 6 * 1 = 6 \quad (hint\ \ln(e) = 1, can\ do\ on\ calculator\ if\ needed)$
$6ln|1| = 6 * 0 = 0\ \ (hint\ \ln(1) = 0, also\ can\ use\ calculator\ if\ needed)$

Step 3: Subtract the results to determine the answer

$\int_1^e 6x^{-1}dx\ = 6 - 0\ = 6$

Answer: $\int_1^e 6x^{-1}dx = 6$

Example: Use the Fundamental Theorem of Calculus to evaluate the definite integral.

$\int_0^1 20x(5x^2-4)^5 dx$

Step 1: Perform the integration.

I borrowed this integration from section 5.2. I attached the work on how to compute the integral on the next few pages. (integration work shown in blue, and takes 2 pages)

$\int_0^1 20x(5x^2-4)^5 dx = \frac{1}{3}(5x^2-4)^6 \Big|_0^1$

Step 2: Evaluate the integral at 1 then at 0

$\frac{1}{3}(5(1)^2-4)^6 = \frac{1}{3}(1)^6 = \frac{1}{3} * 1 = \frac{1}{3}$

$\frac{1}{3}(5(0)^2-4)^6 = \frac{1}{3}(-4)^6 = \frac{1}{3}(4096) = \frac{4096}{3}$

Step 3: Subtract the results to determine the answer

$\int_0^1 20x(5x^2-4)^5 dx = \frac{1}{3} - \frac{4096}{3} = -\frac{4095}{3} = -1365$

$(The\ negative\ indicates\ more\ of\ the\ graph\ is\ beneath\ the\ x - axis\ than\ is\ above.)$

Example: Rewrite a problem that has a parenthesis with an exponent using $u - substitution$, then integrate to find the answer.

$\int 20x(5x^2 - 4)^5 dx$

Rewrite the problem so that the parenthesis is first:

$= \int (5x^2 - 4)^5 20x dx$

Next: $let\ u = inside\ of\ the\ parenthesis$

$let\ u = 5x^2 - 4$

Rewrite the problem so that the "parenthesis is changed to an "u"

$= \int u^5 20x dx$

Next find $\frac{du}{dx}$

$u = 5x^2 - 4$

$\frac{d}{dx} u = \frac{d}{dx} 5x^2 - \frac{d}{dx} 4$

$\frac{du}{dx} = 10x$

Multiply by dx to clear the fraction.

$dx \frac{du}{dx} = 10x dx$

$du = 10x dx$

This is not good enough. I need to replace $20xdx$.

Multiply by 2.
$2du = 2 * 10xdx$
$2du = 20xdx$

Next replace $20xdx \; with \; 2du$

$= \int u^5 20xdx \; = \int u^5 2du \; = \int 2u^5 du$

Rewrite using: $\int af(x)dx = a\int f(x)dx$

$= 2\int u^5 du$
Next integrate: use Power Rule: $\int x^n dx = \frac{1}{n+1}x^{n+1} + C \quad provided \; x \neq -1$

$= 2 * \frac{1}{6}u^6 + C$

$= \frac{1}{3}u^6 + C$

Last change $u \; to \; 5x^2 - 4 \; to \; get \; the \; answer$

$Answer$: $\frac{1}{3}(5x^2 - 4)^6 + C$

Section 5.4 The Fundamental Theorem of Calculus
(Minimum Homework: all odds)

#1-24 Use the Fundamental Theorem of Calculus to evaluate the definite integral.

1) $\int_2^5 (4x-3)dx$

2) $\int_1^6 (2x-5)dx$

1st:

$$\int \big(f(x)-g(x)\big)dx = \int f(x)dx \quad - \int g(x)dx$$

2nd: $\int af(x)dx = a\int f(x)dx$

3rd:

Power Rule:

Integral of a constant Rule: $\int adx \; = \; ax +$
$C \quad (a\ is\ any\ real\ number)$

4th Evaluate the integral at 1 then at 0

5th subtract the results to get the answer

$answer$: $\int_1^6 (2x-5)dx = 10$

3) $\int_3^7 5dx$

4) $\int_2^9 6dx$

1st: Integral of a constant Rule: $\int adx = ax + C \quad (a \text{ is any real number})$

2nd Evaluate the integral at 9 then at 2

3rd subtract the results to get the answer

Answer: $\int_2^9 6dx = 42$

5) $\int_0^3 (4x^3 + 3x^2 - 7)dx$

6) $\int_0^2 (8x^3 - 6x^2 + 2)dx$

1st:

$$\int \big(f(x) + g(x)\big)dx = \int f(x)dx \quad + \int g(x)dx$$

$$\int \big(f(x) - g(x)\big)dx = \int f(x)dx \quad - \int g(x)dx$$

2nd: $\int af(x)dx = a \int f(x)dx$

3rd:

Power Rule:

Integral of a constant Rule:

4th Evaluate the integral at 2 then at 0

5th subtract the results to get the answer

$answer$: $\int_0^2 (8x^3 - 6x^2 + 2)dx = 20$

7) $\int_0^2 3e^x dx$

8) $\int_0^4 2e^x dx$

1st: $\int af(x)dx = a\int f(x)dx$

2nd: "e" Rule $\int e^x dx \quad = e^x + C$

3rd Evaluate the integral at 4 then at 0

4th subtract the results to get the answer

$answer$: $\int_0^4 2e^x dx = 2e^4 - 2$

9) $\int_1^e \frac{3}{x} dx$

10) $\int_1^{e^2} \frac{5}{x} dx$

1st: Rewrite with -1 exponent

2nd: $\int af(x)dx = a\int f(x)dx$

3rd: "ln" Rule: $\begin{cases} \int x^{-1}dx = \ln|x| + C \\ \int \frac{1}{x} dx = \ln|x| + C \end{cases}$

4th Evaluate the integral at e^2 then at 1

5th subtract the results to get the answer

$answer$: $\int_1^{e^2} \frac{5}{x} dx = 10$

11) $\int_1^e 7x^{-1}dx$

12) $\int_1^e 2x^{-1}dx$

1st : $\int af(x)dx = a\int f(x)dx$

2nd : "ln" Rule: $\begin{cases} \int x^{-1}dx = \ln|x| + C \\ \int \frac{1}{x}dx = \ln|x| + C \end{cases}$

3rd Evaluate the integral at e then at 1

4th subtract the results to get the answer

$answer$: $\int_1^e 2x^{-1}dx = 2$

13) $\int_1^2 3(3x+1)^2 dx$

14) $\int_1^2 7(7x-4)^2 dx$ Rewrite the problem so that the parenthesis is first:

Next: $let\ u = inside\ of\ the\ parenthesis$

Rewrite the problem so that the "parenthesis is changed to an "u"

Next find $\frac{du}{dx}$

Multiply by dx to clear the fraction.

Next replace to make problem only have u's

Next integrate: use Power Rule:

Last change $u\ back\ to\ compute\ the\ integral$

Evaluate the integral at 2 then at 1

subtract the results to get the answer$answer$

$$\int_1^2 7(7x-4)^2 dx = \frac{973}{3}$$

15) $\int_1^2 9(3x+1)^2\,dx$

16) $\int_1^2 14(7x-4)^2\,dx$

Rewrite the problem so that the parenthesis is first:

Next: $let\ u = inside\ of\ the\ parenthesis$

Rewrite the problem so that the “parenthesis is changed to an "u"

Next find $\frac{du}{dx}$

Multiply by dx to clear the fraction.

Not good enough, multiply to make a perfect match

Next replace to make problem only have u’s

Next integrate: use Power Rule:

Last change $u\ back\ to\ compute\ the\ integral$

Evaluate the integral at 2 then at 1

subtract the results to get the answer

$answer\text{:}\ \int_1^2 14(7x-4)^2\,dx = \frac{1946}{3}$

17) $\int_{-2}^{4}(2x)\,(x^2-1)^2dx$

18) $\int_{-2}^{1}(2x-4)(x^2-4x+1)^2dx$

Rewrite the problem so that the parenthesis is first:

Next: $let\ u = inside\ of\ the\ parenthesis$

Rewrite the problem so that the “parenthesis is changed to an "u"

Next find $\frac{du}{dx}$

Multiply by dx to clear the fraction.

Next replace to make problem only have u’s

Next integrate: use Power Rule:

Last change $u\ back\ to\ compute\ the\ integral$

Evaluate the integral at 1 then at -2

subtract the results to get the answer

$answer$: $\int_{-2}^{1}(2x-4)(x^2-4x+1)^2dx = -735$

19) $\int_{-2}^{4}(6x)\,(x^2-1)^2dx$

20) $\int_{-2}^{1}(6x-12)(x^2-4x+1)^2dx$

Rewrite the problem so that the parenthesis is first:

Next: $let\ u = inside\ of\ the\ parenthesis$

Rewrite the problem so that the “parenthesis is changed to an "u"

Next find $\frac{du}{dx}$

Multiply by dx to clear the fraction.

Not good enough, multiply to make a perfect match

Next replace to make problem only have u’s

Next integrate: use

Last change $u\ back\ to\ compute\ the\ integral$

Evaluate the integral at 1 then at -2

subtract the results to get the answer

$answer$: $\int_{-2}^{1}(6x-12)(x^2-4x+1)^2dx = -2205$

21) $\int_0^1 3x^2 e^{x^3} dx$

22) $\int_0^1 2xe^{x^2}\, dx$

Rewrite the problem so that the “e” is written first:

Next: $let\ u = exponent\ of\ the\ e$

Rewrite the problem so that the exponent is changed to an "u"

Next find $\frac{du}{dx}$

Multiply by dx to clear the fraction.

Next replace to make problem only have u’s

Next integrate: "e" Rule $\int e^x dx \ = e^x + C$

Last change $u\ back\ to\ calculate\ the\ integral$

Evaluate the integral at 1 then at 0

subtract the results to get the answer

answer: $\int_0^1 2xe^{x^2}\, dx = e - 1$

23) $\int_0^1 6x^2 e^{x^3} dx$

24) $\int_0^1 8xe^{x^2} dx$

Rewrite the problem so that the "e" is written first:

Next: $let\ u = exponent\ of\ the\ e$

Rewrite the problem so that the exponent is changed to an "u"

Next find $\frac{du}{dx}$

Multiply by dx to clear the fraction.

Not good enough, multiply to make a perfect match

Next replace to make problem only have u's

Next integrate: "e" Rule $\int e^x dx \ = e^x + C$

Last change $u\ back\ to\ calculate\ the\ integral$

Evaluate the integral at 1 then at 0

subtract the results to get the answer

$answer$: $\int_0^1 8xe^{x^2} dx = 4e - 4$